# Kac-Moody Lie Algebras and Flag Varieties

Lecture 1:

Organisation:

- 2 lectures/week
- Oral exam (date TBD)
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## Contents:

0. Semisimple Lie Algebras · Recall
1. Kac-Moody Algebras
2. Kac-Moody Groups and Flag Varieties

## Prerequisites:

- semi-simple Lie algebras + f.d rep's.
- basic AG: basics on complex proj. varieties

<u>Sources:</u>

- Infinite Dimensional Lie Algebras, Kac
- Infinite Dimensional Lie Algebras, Wakimoto
- Kac-Moody Groups, their flag var. and rep.th., Kumar

⋮

⇝ see website

# 0. Semisimple Lie Algebras: Recall

**Global Assumption:** Everything $/ \mathbb{C}$

## 0.1. Lie Algebras

**Def.** A Lie algebra $\mathfrak{g}$ is a vector space $/\mathbb{C}$ with a Lie bracket $[\,,\,] : \Lambda^2 \mathfrak{g} \to \mathfrak{g}$, s.t.

$$\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x,y]$$

is a derivation for each $x$, that is,

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)] \qquad \square$$

**Example** (1) Let $A$ be an associative algebra.

Then $[x,y] = xy - yx$ turns $A$ into a Lie algebra.

(2) For $V$ a v.s. and $A = \mathrm{End}(V)$, we obtain

$$\mathfrak{gl}(V) = \mathrm{End}(V) + [\,,\,]$$

(3) Let $G$ be a complex Lie/algebraic group. Then

$$\mathfrak{g} = \mathrm{Lie}(G) = T_e X$$

is a Lie algebra

(5) The set of derivations $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra.

(6) $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n, \ldots$

(6) The $2n+1$-dim. Lie algebra $\mathfrak{g}$, spanned by

$u_1, \dots, u_n, v_1, \dots, v_n, z$ with bracket defined by

$$[u_i, v_j] = \delta_{ij} z \quad , \quad [y, z] = 0$$

$$(\Rightarrow [u_i, u_j] = [v_i, v_j] = 0)$$

is called the <u>Heisenberg algebra</u>

(7) The most important example:

$$sl_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} = \left\langle e = \begin{pmatrix} & 1 \\ & \end{pmatrix}, h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, f = \begin{pmatrix} & \\ 1 & \end{pmatrix} \right\rangle \subset gl(\mathbb{C}^2)$$

It has relations: $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$

$\square$

**Def. B**: A representation $V$ of $\mathfrak{g}$ is a map (of Lie algebras), $\mathfrak{g} \to \mathfrak{gl}(V)$ $\square$

**Example B** (1) $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $x \mapsto \mathrm{ad}(x)$ defines the adjoint representation.

We have $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{der}(\mathfrak{g})$.

(2) For $\mathfrak{sl}_2(\mathbb{C})$ we obtain (with the basis $e, h, f$)

$$\mathrm{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad \mathrm{ad}(h) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix} \quad \mathrm{ad}(f) = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & 2 & 0 \end{pmatrix}$$

(3) Let $\mathfrak{g} = \langle u, v, z \rangle$ be the 3-dim Heisenberg algebra. Then we obtain a faithful representation:

$$u \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Generalize this from 3 to $2n+1$)

(4) For the $2n+1$-dim. Heisenberg algebra, we obtain the <u>Schrödinger representation</u> on $\mathbb{C}[x_1, .., x_n]$ via

$$u_i \mapsto \frac{\partial}{\partial x_i}, \quad v_i \mapsto x_i, \quad z \mapsto \mathrm{id} \qquad \square$$

## 0.2 Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra.

<u>Def</u>: The <u>universal enveloping algebra</u> $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra defined by

$$T^\bullet V \big/ \left( x \otimes y - y \otimes x = [x,y] \right) \qquad \square$$

<u>Rem</u>: We omit the $\otimes$ in the notation for elements in $T^\bullet V$

<u>Prop</u>: $U(\mathfrak{g})$ is determined by the universal property

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & U(\mathfrak{g}) \arrow[d, dashed, "\exists !"] \\ & A \end{tikzcd}$$

Lie alg. map

unital associative algebra $\qquad \square$

Thm: (Poincaré - Birkhoff - Witt) Denote by $U_n(g) \subset U(g)$ the image of $\bigoplus_{i \leq n} T^i(g)$ under $T^\bullet g \twoheadrightarrow U(g)$. Then

(1) $U_n(g) \subset U_m(g)$ for $n \leq m$

(2) $U_n(g) U_m(g) \subset U_{n+m}(g) \quad \forall n, m$

(3) $gr\, U(g) = \bigoplus_n U_n(g)/U_{n-1}(g)$ is a commutative graded algebra

(4) The natural map $Sym(g) \to gr\, U(g)$ is an isom. of graded algebras $\square$

The Thm. implies that as vector space, $U(g)$ and $Sym(g)$ behave the same.

## 0.3 Automorphisms

Let $\dim \mathfrak{g} < \infty$

<u>Def</u> The group of (inner) automorphism of $\mathfrak{g}$ is

$$\begin{array}{ccccc} \mathrm{Int}(\mathfrak{g}) & \subset & \mathrm{Aut}(\mathfrak{g}) & \subset & GL(\mathfrak{g}) \\ \| & & \uparrow & & \\ \langle e^{\mathrm{ad}(x)} \mid x \in \mathfrak{g} \rangle & & \text{Lie algebra autom.} & & \end{array}$$

$\square$

<u>Prop</u>: $\mathrm{Int}(\mathfrak{g}), \mathrm{Aut}(\mathfrak{g})$ are Lie groups with Lie algebras

$$\begin{array}{ccccc} \mathrm{Lie}(\mathrm{Int}(\mathfrak{g})) & \to & \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) & \to & \mathfrak{gl}(\mathfrak{g}) \\ \wr\| & & \wr\| & & \\ \mathfrak{g}/\mathfrak{z} & \xrightarrow{\mathrm{ad}} & \mathrm{Der}(\mathfrak{g}) & & \end{array}$$

where $\mathfrak{z} = \ker(\mathrm{ad})$

$\square$

Remark: (1) The exponential $e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j$ for $X \in End(V)$

works well when either

(a) $\dim V < \infty$

(b) $X$ is locally nilpotent, that is $\forall v \in V \; \exists n \geq 0$, s.t.

$X^n v = 0$

(2) Under this conditions $e^{ad(X)} = Ad(e^X)$ and similar

standard formulas hold $\square$

Lecture 2

## 0.4. Representation theory of $\mathfrak{sl}_2$

We recall the rep.-th. of

$$\mathfrak{sl}_2(\mathbb{C}) = \left\langle e = \begin{pmatrix} & 1 \\ & \end{pmatrix}, h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, f = \begin{pmatrix} & \\ 1 & \end{pmatrix} \right\rangle$$

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

<u>Lemma</u>: The following relations hold in $U(\mathfrak{sl}_2(\mathbb{C}))$

(1) $[h, e^k] = 2k\, e^k, \qquad [h, f^k] = -2k\, f^k$

(2) $[e, f^k] = k\, f^{k-1}(h+1-k) = k\,(h+k-1)\, f^{k-1}$

$\phantom{(2)}$ $[f, e^k] = -k\, e^{k-1}(h+k-1) = -k\,(h+1-k)\, e^{k-1}$

<u>Proof</u>: (1) $[h, e^k] = [h,e]e^{k-1} + e\,[h, e^{k-1}]$

$$= \underset{\text{relation}}{2e}\, e^{k-1} + e\, \underset{\text{induction}}{2(k-1)\, e^{k-1}} = 2k\, e^k$$

(2) $[e, f^k] = [e,f] f^{k-1} + f \cdot [e, f^{k-1}]$

$$= h f^{k-1} + f (k-1)(h+k-2) f^{k-2}$$

$$\overset{fh = hf+2f}{=} h f^{k-1} + (k-1) h f^{k-1} + (k-1) 2 f^{k-1} + (k-1)(k-2) f^{k-1}$$

$$= k(h+k-1) f^{k-1}$$

The other equations are analogons $\square$

<u>Prop.</u> Let $V$ be a $\mathfrak{sl}_2$-rep. with $v \in V$, $\lambda \in \mathbb{C}$ such that $hv = \lambda v$. Let $v_j = \frac{1}{j!} f^j v$.

Then

$$h v_j = (\lambda - 2j) v_j$$

If $ev = 0$, then we have

$$ev_j = (\lambda - j + 1)\, v_{j-1}$$

<u>Proof</u>: Use Lemma (1) and (2) $\square$

<u>Rem</u>: (1) If $hv = \lambda v$ and $ev = 0$, $v$ is called a highest weight vector (of weight $\lambda$)

(2) One can visualise this as:

$$\begin{tikzcd}
\cdots & v_{j+1} \arrow[loop above, "\lambda-2(j+1)"] \arrow[r, bend left, "\lambda-j"] & v_j \arrow[loop above, "\lambda-2j"] \arrow[l, bend left, "j"] & \cdots \arrow[r, bend left, "\lambda-2"] & v_2 \arrow[loop above, "\lambda-4"] \arrow[l, bend left, "3"] \arrow[r, bend left, "\lambda-1"] & v_1 \arrow[loop above, "\lambda-2"] \arrow[l, bend left, "2"] \arrow[r, bend left, "\lambda"] & v_0 \arrow[loop above, "\lambda"] \arrow[l, bend left, "1"]
\end{tikzcd}$$

where $e : \longrightarrow$, $h : \circlearrowright$, $f : \longleftarrow$

<u>Def A</u> Let $\lambda \in \mathbb{C}$, $\mathfrak{b} = \langle h, e \rangle \subset sl_2$. Then

$$M(\lambda) = U(sl_2) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

is the <u>Verma module</u> (or <u>universal highest weight module</u>) for $sl_2$ of highest weight $\lambda$. Here $\mathfrak{b}$ acts on $\mathbb{C}_\lambda = \mathbb{C}$ via $\quad e v = 0, \; h v = \lambda v \;\; \forall v \in \mathbb{C}_\lambda$ $\square$

<u>Prop. B</u>: Let $n^- = \langle f \rangle \subset sl_2$, $\lambda \in \mathbb{C}$ and $v^+ \in \mathbb{C}_\lambda \setminus \{0\}$.

(1) $\mathbb{C}[f] = \mathrm{Sym}(n^-) = \mathcal{U}(n^-) \longrightarrow M(\lambda), \quad 1 \mapsto v^+$

is an isomorphism of $\mathcal{U}(n^-)$-modules. In particular $\{ v_j = \frac{1}{j!} f^j v^+ \}_{j \geq 0}$ is a basis of $M(\lambda)$

(2) Let $v_0$ be a highest weight vector in $V$ of weight $\lambda$. Then there is a unique map

$$M(\lambda) \longrightarrow V, \quad v^+ \longmapsto v_0$$

<u>Proof</u> Exercise $\square$

<u>Thm:</u> Let $\lambda \in \mathbb{C}$. Then

(1) $M(\lambda)$ is irreducible $\Leftrightarrow$ $\lambda \notin \mathbb{Z}_{\geq 0}$

(2) Let $\lambda \in \mathbb{Z}_{\geq 0}$, then there is an injective map

$$\begin{aligned} M(-\lambda-2) &\hookrightarrow M(\lambda) \twoheadrightarrow L(\lambda) \\ w^+ &\longmapsto \tfrac{1}{(\lambda+1)!} f^{(\lambda+1)} v^+ \end{aligned}$$

and the quotient is a irr. f.d. representation $L(\lambda)$

of dimension $\lambda+1$. Here $w^+, v^+$ are the highest weight vectors of $M(-\lambda-2)$ and $M(\lambda)$, respectively.

(3) Let $L$ be a simple, f.d. rep. of $sl_2(\mathbb{C})$. Then there is a (unique up to scalar) highest weight vector $v_0 \in L$. Then $v_0$ has weight $\lambda \in \mathbb{Z}_{\geq 0}$ and the natural map $M(\lambda) \to L(\lambda)$ yield an iso. $L(\lambda) \xrightarrow{\sim} L$ $\square$

__Pf__: Denote a basis of $M(\lambda)$ as in Prop. B. Then by Prop A,

(*) $e v_{j+1} = (\lambda - j) v_j = 0 \iff \lambda = j$

(1) "$\Leftarrow$" Assume that $\lambda \notin \mathbb{Z}_{\geq 0}$ and let $x \in M(\lambda) \setminus \{0\}$.

Let $k \in \mathbb{Z}$, s.t. $x = a_k v_k + \sum_{j<k} a_j v_j$ with

$a_k \neq 0$. Then by (*) $e^k x = a v^+$ where $a \neq 0$

Since $v^+$ generates $M(\lambda)$, so does $x$. Hence

$M(\lambda)$ is irreducible

(2) + (1) "$\Rightarrow$" If $\lambda \in \mathbb{Z}_{\geq 0}$, by (*) $v_{\lambda+1}$

highest weight vector of (by Prop A) weight

$\lambda - 2(\lambda+1) = -\lambda - 2$. Hence, there is a unique map

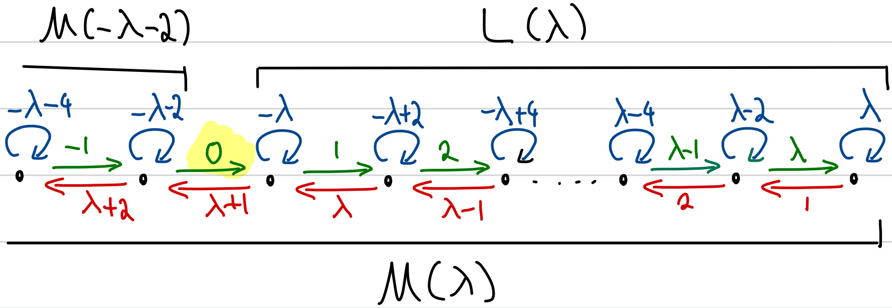
$M(-\lambda-2) \longrightarrow M(\lambda)$, $w^+ \mapsto v_{\lambda+1}$.

Since $-\lambda-2 \notin \mathbb{Z}_{\geq 0}$, $M(-\lambda-2)$ is irreducible and the map is hence injective.

By a similar argument as in the proof of (1) "$\Leftarrow$", the quotient $L(\lambda) = M(\lambda)/M(-\lambda-2)$ is irreducible, and $L(\lambda)$ has basis $\bar{v}_0, \dots, \bar{v}_\lambda$.

(3) Exercise. Hint: decompose $L$ in eigenspaces of $h$. $\square$

<u>Sketch</u> For $\lambda \in \mathbb{Z}_{\geq 0}$, we obtain:



Example: (1) $L(0) = \mathbb{C}_0$ = trivial

$L(1) = \mathbb{C}^2$ = fundamental

$L(2) = \mathfrak{sl}_2$ = adjoint

In general $L(\lambda) = \mathrm{Sym}^\lambda(\mathbb{C}^2) = k[x,y]_\lambda$

homogeneous polynomials of degree $\lambda$ $\triangleright$

Lecture 3

We will study Lie algebras via copies of $\mathfrak{sl}_2$'s they contain:

**Def B**: Let $\mathfrak{g}$ be a Lie algebra. A choice of elements

$e, h, f \in \mathfrak{g}$ with $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$

is called an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$

## 0.5 Killing form

Let $\mathfrak{g}$ be a Lie algebra

<u>Def</u>: (1) A symmetric form $B: \mathfrak{g}\otimes\mathfrak{g} \to k$ is called $\mathfrak{g}$-<u>invariant</u> if

$$B([x,y],z) = B(x,[y,z]) \quad \forall x,y,z \in \mathfrak{g}$$

(2) If $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is a f.d. rep. of $\mathfrak{g}$, we denote

$$(x,y)_V = \mathrm{tr}(\rho(x)\rho(y))$$

If $\rho = \mathrm{ad}$, $(\,,\,) = (\,,\,)_V$ is the <u>Killing form</u>

(3) Assume that $\mathfrak{g}$ is f.d. and $B_V$ as in (2) is non-deg. Pick a pair of dual bases $\{x_i\}$, $\{x^i\}$ of $\mathfrak{g}$, that is, $(x_i, x^j)_V = \delta_{ij}$. Then

the <u>Casimir operator</u> is defined by

$$C_V = \sum x_i x^i$$

<u>Lemma</u> (1) $( \, , \, )_V$ is a $\mathfrak{g}$-invariant form.

(2) $C_V$ does not depend on the choice of basis.

(3) $C_V \in Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$

<u>Proof</u>: Omitted $\square$

<u>Example</u> Let $\mathfrak{g} = \mathfrak{sl}_2$, $B(X,Y) = tr(XY)$, then

we get dual bases $\{e, h, f\}$, $\{f, \frac{1}{2}h, e\}$ and

the Casimir operator

$$C = ef + \frac{1}{2}h^2 + fe = \frac{1}{2}h^2 + h + 2fe$$

By Schur's Lemma $C$ act by a scalar on $L(\lambda)$. Let $v^+ \in L(\lambda)$ the highest weight vector, then $C v^+ = (\frac{1}{2}h^2 + h + fe)\, v^+ = (\frac{1}{2}\lambda^2 + \lambda) \cdot v^+$

Since $\lambda \mapsto \frac{1}{2}\lambda^2 + \lambda$ is injective for $\lambda \in \mathbb{Z}_{\geq 0}$, we can decompose f.d. $sl_2$-rep's in isotypic components using eigenspaces of $C$ $\square$

## 0.6 Semisimple Lie Algebras – Definitions

Def/Thm: A f.d. Lie algebra $\mathfrak{g}$ is called <u>semisimple</u> if (one of) the following equivalent statements is fulfilled:

(1) $\mathfrak{g}$ has no Abelian ideals

(2) $\mathfrak{g}$ has no solvable ideals

(3) The Killing form $( \, , \, )$ is non-deg.

(4) $\mathfrak{g}$ is a sum of simple Lie algebras

Moreover $\mathfrak{g}$ is called <u>reductive</u> if $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ with $\mathfrak{g}'$ semisimple and $\mathfrak{z}$ Abelian

Proof: Idea: $\mathrm{rad}((\,,\,)) \subset \mathrm{rad}(\mathfrak{g})$ ....

radical of Killing form — max solvable ideal in $\mathfrak{g}$ $\square$

Thm: Let $\mathfrak{g}$ be a f.d. Lie algebra. Then

$\mathfrak{g}$ reductive $\iff$ every f.d. rep. is a sum of irreducibles

Proof: Sketch:

"$\Leftarrow$" Decompose $V = \mathfrak{g}$ in irreducibles ...

"$\Rightarrow$" It suffices to show: $W \mapsto \mathrm{Hom}_{\mathfrak{g}}(V, -)$ is exact for $V, W$ f.d. rep's.

To see this, use

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}(V, W)^{\mathfrak{g}}.$$

Since Hom is exact, it suffices to show that

$$V \mapsto V^{\mathfrak{g}} = \mathrm{Hom}(\mathrm{triv}, V)$$

is exact, or that $\mathrm{Ext}^1(\mathrm{triv}, V) = 0$, or that every s.e.s. of the form

$$0 \to V \to E \to \mathrm{triv} \to 0$$

splits. If $V \neq \mathrm{triv}$ is irreducible, there is a splitting $E = V \oplus \ker(C_V)$, where $\ker(C_V) \cong \mathrm{triv}$ and $C_V$ is the Casimir of $V$.

If $V = \mathrm{triv}$, $\rho_E(\mathfrak{g}) \subset \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ is solvable. Since $\mathfrak{g}$ is semisimple, $\rho_E(\mathfrak{g}) = 0$, so $E = \mathrm{triv}^{\oplus 2}$.

For $V$ not-irreducible, use induction $\square$

## 0.7. Cartan subalgebras, roots, coroots and $\mathfrak{sl}_2$'s

Let $\mathfrak{g}$ be a semisimple Lie algebra.

<u>Def A</u>: We call $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, if (1) $\mathfrak{h}$ is a maximal Abelian subalgebra

(2) $\mathrm{ad}(h)$ is diagonalizable $\forall h \in \mathfrak{h}$ $\square$

<u>Thm</u>: All Cartan subalgebras are conjugate wrt. $\mathrm{Int}(\mathfrak{g})$.

<u>Proof</u>: Omitted $\square$

<u>Example</u>: $\mathfrak{h} = \mathfrak{sl}_n \cap \mathrm{diag} \subset \mathfrak{sl}_n$ is a Cartan subalgebra. $\square$

From now, choose a Cartan $\mathfrak{h} \subset \mathfrak{g}$

Def B: (1) For a rep. $V$ of of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$,

we call

$$V_\lambda = \{ v \in V \mid hv = \lambda(h) v \ \forall h \in \mathfrak{h} \}$$

the weight space of $V$ (wrt. $\lambda$). If $V_\lambda \neq 0$,

we call $\lambda$ a weight of $V$

(2) We call $0 \neq \alpha \in \mathfrak{h}^*$ a root of $\mathfrak{g}$ if it

is a weight of the adjoint rep. $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$

and denote by $\Delta \subset \mathfrak{h}^*$ the set of roots

(3) We denote by

$$\mathfrak{g} = \underset{\mathfrak{h}}{\underset{\parallel}{\mathfrak{g}_0}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

the weight space decomposition of $\mathfrak{g}$ $\square$

Exercise: Compute this for $\mathfrak{sl}_n$ (MUST DO) $\square$

For $\alpha, \beta \in \Delta$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and $H \in \mathfrak{h}$, we get

$$[H,[X,Y]] = (\alpha+\beta)(H)[X,Y] \quad \text{and}$$

$$([X,Y],H) = -\alpha(H)(X,Y) = \beta(H)(X,Y)$$

Some immediate consequences:

Lemma: Let $\alpha, \beta \in \Delta$.

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$

(3) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\beta \neq -\alpha$

(4) $(\mathfrak{g}_\alpha, \mathfrak{h}) = 0$

(5) $( \, , \, )_{\mathfrak{h}\times\mathfrak{h}}$ is non-deg.

(6) $( \, , \, )_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-deg pairing

(6) $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$

(7) $\bigcap_{\alpha\in\Delta} \ker\alpha \subset \mathfrak{z}(\mathfrak{g}) = 0$

(8) $\mathfrak{h}^* = \langle \alpha \in \Delta \rangle_{\mathbb{C}}$ $\square$

Our next goal is to construct for each $\alpha\in\Delta$

a copy $\mathfrak{s}_\alpha \subset \mathfrak{g}$ of $\mathfrak{sl}_2$ with a basis $\{\underset{\mathfrak{g}_\alpha}{\underset{\cap}{e_\alpha}}, h_\alpha, \underset{\mathfrak{g}_{-\alpha}}{\underset{\cap}{f_\alpha}}\}$.

For this

$$V_{\alpha,\beta} = \bigoplus \mathfrak{g}_{\alpha+i\beta}$$

With a bit more work, one can show:

<u>Thm B:</u> For each $\alpha \in \Delta$, let

$$s_\alpha = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}.$$

Then there is an isomorphism $\mathfrak{sl}_2 \cong s_\alpha$ and an $\mathfrak{sl}_2$-triple

$e_\alpha, h_\alpha = \alpha^\vee, f_\alpha \in s_\alpha$, s.t., $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$, $\mathfrak{g}_{-\alpha} = \mathbb{C} e_{-\alpha}$.

Moreover, $h_\alpha$ is uniquely determined by $\alpha(h_\alpha) = 2$.

<u>Proof:</u> Sketch: Pick $X_\alpha \in \mathfrak{g}_\alpha$, $X_\alpha \in \mathfrak{g}_{-\alpha}$, s.t.

$(X_\alpha, X_{-\alpha}) = 1$. Let $H_\alpha = [X_\alpha, X_{-\alpha}]$. Show

that $\alpha(H_\alpha) \neq 0$ and set $h_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha$, $e_\alpha = X_\alpha$

and $f_\alpha = \frac{2}{\alpha(H_\alpha)} X_{-\alpha}$ $\square$

__Def B__: The elements $\alpha^\vee = h_\alpha \in \mathfrak{h}$ are called __coroots__. The set of coroots is denoted by

$$\Delta^\vee = \{\alpha^\vee \in \mathfrak{h}^* \mid \alpha \in \Delta\} \qquad \square$$

Lecture 4

## 0.9. Adjoint action of $\mathfrak{s}_\alpha$ on $\mathfrak{g}$.

For roots $\alpha, \beta \in \Delta$, we consider

$$V_{\alpha,\beta} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha} \subset \mathfrak{g}.$$

Now $V_{\alpha,\beta}$ is a f.d. representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$

Let $p, q$ be minimal, resp. maximal, s.t.

$\mathfrak{g}_{\beta + p\alpha} \neq 0$ and $\mathfrak{g}_{\beta + q\alpha} \neq 0$.

In particular $p \leq 0 \leq q$

Prop A: (1) $p+q = -\beta(h_\alpha) \in \mathbb{Z}$

(2) $\{\beta + j\alpha \mid j \in \mathbb{Z}\} \cap (\Delta \cup \{0\}) = \{\beta + j\alpha \mid p \leq j \leq q\}$

(3) $\beta + (p+q)\alpha = \beta - \beta(h_\alpha) \in \Delta$

(4) If $\alpha \pm \beta \neq 0$, then

(a) if $\alpha + \beta \notin \Delta$, $\beta(h_\alpha) \geq 0$

(b) if $\alpha - \beta \notin \Delta$, $\beta(h_\alpha) \leq 0$

(5) $[s_\alpha, s_\beta] = 0 \iff \beta(h_\alpha) = 0 \iff \alpha(h_\beta) = 0$

(6) If $\alpha + \beta \neq 0$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

(7) $\mathbb{Z}\alpha \cap \Delta = \{\pm\alpha\}$

Proof: (1), (2), (3) $h_\alpha \in \mathfrak{s}_\alpha$ acts on $\mathfrak{g}_{\beta+i\alpha}$ via

$$(\beta+i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i.$$

So, we obtain a $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ rep. with $h_\alpha$ action

$$\begin{array}{ccccc} \mathfrak{g}_{\beta+p\alpha} & \cdots & \mathfrak{g}_{\beta+i\alpha} & \cdots & \mathfrak{g}_{\beta+q\alpha} \\ \circlearrowright & & & & \circlearrowright \\ \beta(h_\alpha)+2p & & & & \beta(h_\alpha)+2q \end{array}$$

From f.d. rep. th. of $\mathfrak{sl}_2$, see 0.4, we know that the extremal weights differ by a sign, so that

$$\beta(h_\alpha) + 2p = -(\beta(h_\alpha) + 2q)$$

and hence

$$\beta(h_\alpha) = -(p+q) \in \mathbb{Z}.$$

Moreover, $\mathfrak{g}_{\beta+i\alpha} \neq 0$ for all $p \leq i \leq q$.

In particular for $i = p+q$.

(4)-(7) : Exercise $\square$

<u>Def</u>: We call the sequence

$$\beta + p\alpha, \dots\dots, \beta + q\alpha$$

the $\alpha$-string through $\beta$ $\square$

# 0.10. Root system and Cartan matrix

Let $\mathfrak{h} \subset \mathfrak{g}$ be a semisimple Lie algebra with Cartan.

So far, we defined

(1) Roots $\alpha \in \Delta \subset \mathfrak{h}^*$

(2) Coroots $h_\alpha = \alpha^\vee \in \Delta^\vee \subset \mathfrak{h}$

(3) A bijection $\Delta \to \Delta^\vee$, $\alpha \mapsto \alpha^\vee$

We denote the natural pairing

$$\langle , \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}.$$

and for $R \subset \mathbb{C}$ a subring

$$\mathfrak{h}_R = \sum_{\alpha^\vee \in \Delta^\vee} A\alpha^\vee \quad , \quad \mathfrak{h}^*_R = \sum_{\alpha \in \Delta} A\alpha$$

By 0.9. Prop (1), we have

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) \in \mathbb{Z} \qquad \forall \alpha, \beta \in \Delta$$

So the pairing descends, for $R \subset \mathbb{C}$, to

$$\langle\ ,\ \rangle : \mathfrak{h}_R \times \mathfrak{h}_R^* \to R.$$

We now collect all the properties of roots and coroots in a convenient framework.

Def: An abstract root system is a tuple

$$(V, \Delta, \Delta^\vee, (\ )^\vee),$$ such that

(1) $V$ is a real vector space, generated by $\Delta$.

(2) $\Delta \subset V$, $\Delta^\vee \subset V^\vee$ are finite subsets and

$(\ )^\vee$: — " a bijection

(3) For $\alpha, \beta \in \Delta$, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ and $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $r_\alpha : V \to V$, $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$

permutes $\Delta$.

(5) If $\alpha$ and $c\alpha \in \Delta$, then $c = \pm 1$

Thm A $(\mathfrak{h}^*_{\mathbb{R}}, \Delta, \Delta^\vee, (\ )^\vee)$ is an abstract root system

Proof: (1) (2) Clear

(3) 0.7 Thm B + 0.9 Prop (1)

(4) 0.9 Prop (3)

(5) 0.9 Prop (7) $\square$

Remark:

In particular, there is a notion of positive, simple roots, Weyl group, ... ($\rightsquigarrow$ Bourbaki Lie Groups and Lie Algebras - Chapters 4-6) :

(1) Choose $\gamma \in \mathfrak{h}_{\mathbb{R}}$, s.t., $\langle \alpha, \lambda \rangle \neq 0 \quad \forall \alpha \in \Delta$.

Then define $\Delta_{\pm} = \{\alpha \in \Delta \mid \pm\langle \alpha, \gamma \rangle \geq 0\}$

and $\Delta_{\pm}^{\vee} = (\Delta_{\pm})^{\vee}$. The elents in $\Delta_{\pm}$ are

called positive/negative roots.

(2) A root $\alpha \in \Delta_{+}$ is called simple if it is

indecomposable, that is,

$$\alpha \cap \sum_{\substack{\beta \in \Delta^{+} \\ \beta \neq \alpha}} \mathbb{Z}_{\geq 0} \beta = \emptyset$$

The set of simple roots is denoted by $\Pi$.

Simple roots form a basis of $\mathfrak{h}^{*}$

(3) The <u>Weyl group</u> is defined by

$$W = \langle r_\alpha \mid \alpha \in \Delta \rangle = \langle r_\alpha \mid \alpha \in \Pi \rangle \subset GL(\mathfrak{h}_{\mathbb{Z}}^*)$$

. . . . $\square$

We will now take a closer look at root strings:

<u>Thm B</u>: Let $\alpha, \beta \in \Pi$ be simple roots. Then

(1) The $\alpha$-string through $\beta$ is of the form

$$\beta, \beta+\alpha, \ldots, \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

and $\langle \beta, \alpha^\vee \rangle \leq 0$

(2) $U(s_\alpha)\, \mathfrak{g}_\alpha$ is the f.d. $sl_2$-rep. of highest weight $-\langle \beta, \alpha^\vee \rangle$ and dimension $1 - \langle \beta, \alpha^\vee \rangle$

in particular $\mathrm{ad}(\mathfrak{g}_\alpha)^{1-\langle\beta,\alpha^\vee\rangle}\mathfrak{g}_\beta = 0$.

Proof: (1) Claim: $\alpha-\beta \notin \Delta$:

Assume $\alpha-\beta=\gamma\in\Delta$. If $\gamma\in\Delta^+$, $\beta+\gamma=\alpha$ $\lightning$ since $\alpha$ indecomposable. If $\gamma\in\Delta^-$, $-\gamma\in\Delta^+$ and $\alpha+(-\gamma)=\beta$ $\lightning$ since $\beta$ indecomposable //claim.

Hence, the root string is of the form

$$\beta+p\alpha, \dots, \beta+q\alpha$$

with $p=0$. By 0.9 Prop (1), $q=p+q=-\beta(h_\alpha)=-\langle\beta,\alpha^\vee\rangle$. Moreover $-\langle\beta,\alpha^\vee\rangle=q\geq 0$.

(2) Exercise in $\mathfrak{sl}_2$-rep. theory $\square$

## 0.11 The Killing form on the Cartan.

Recall the proof of 0.7. Thm B. For $\alpha \in \Delta$, we picked

$X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, s.t. $(X_\alpha, X_{-\alpha}) = 1$ and set

$H_\alpha = [X_\alpha, X_{-\alpha}]$. Then

$$(H_\alpha, H) = ([X_\alpha, X_{-\alpha}], H) = \alpha(H)\,(X_\alpha, X_{-\alpha}) = \alpha(H).$$

Moreover, $h_\alpha = \dfrac{2H_\alpha}{(H_\alpha, H_\alpha)}$

So, when identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ via the non-deg. form

$( , )$ $\alpha$ corresponds to $H_\alpha$. Note that we

can also transport $( , )$ to a form on $\mathfrak{h}^*$, such that

$$(\alpha, \beta) = (H_\alpha, H_\beta).$$

Prop: Let $\alpha, \beta \in \Delta$

(1) $\langle \beta, \alpha^\vee \rangle = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$

(2) $(\alpha, \beta) \in \mathbb{Q}$

(3) $(\alpha, \alpha) > 0$

(4) $( \, , \, )_{h_\mathbb{R} \times h_\mathbb{R}}$ is positive definite.

Proof: (1) We have

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) = \beta\left(\frac{2H_\alpha}{(H_\alpha, H_\alpha)}\right) = \frac{2\beta(H_\alpha)}{(H_\alpha, H_\alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

(2) Consider the $s_\alpha$-module $V = V_{\alpha, \beta} = \bigoplus \mathfrak{g}_{\beta + i\alpha}$.

Since $H_\alpha = [X_\alpha, X_{-\alpha}]$, $\mathrm{tr}_V(\mathrm{ad}(H_\alpha)) = 0$. On the

other hand

(2)-(4) Consider for $H, H' \in \mathfrak{h}$

$$(H, H') = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(H)\,\mathrm{ad}(H')) = \sum_{\beta \in \Delta} \beta(H)\beta(H').$$

Then, for $H = H' = H_\alpha$, we get

$$(H_\alpha, H_\alpha) = \sum_{\beta \in \Delta} \beta(H_\alpha)^2$$

Hence, we obtain

$$1 = (H_\alpha, H_\alpha) \sum_{\beta \in \Delta} \left( \frac{\beta(H_\alpha)}{(H_\alpha, H_\alpha)} \right)^2$$

$$= (H_\alpha, H_\alpha) \underbrace{\sum \left( \frac{1}{2} \langle \beta, \alpha^\vee \rangle \right)^2}_{\in \mathbb{Q}_{>0}}$$

So $(H_\alpha, H_\alpha) = (\alpha, \alpha) \in \mathbb{Q}_{>0}$. Hence

$$(\alpha, \beta) = \frac{1}{2} \langle \beta, \alpha^\vee \rangle (\alpha, \alpha) \in \mathbb{Q}.$$

Moreover, we see that $h_\alpha$ is a rational multiple of $H_\alpha$. Hence, for $\mathbb{Q} \subset R \subset \mathbb{C}$

$$\mathfrak{h}_R = \sum R h_\alpha = \sum R H_\alpha .$$

To see that $( \, , \, )_{\mathfrak{h}_\mathbb{R} \times \mathfrak{h}_\mathbb{R}}$ is pos. def., let $H \in \mathfrak{h}_\mathbb{R}$. Then, as above

$$(H, H) = \sum_{\beta \in \Delta} \underbrace{(\underbrace{\beta(H)}_{\in \mathbb{R}})^2}_{\mathbb{R}_{\geq 0}} \geq 0 .$$

Since $( \, , \, )$ is non-deg., the statement follows $\square$

Lecture 5

## 0.11 Cartan matrix, Dynkin diagram and Classification

<u>Defn</u>: An $n\times n$-matrix $A=(a_{ij})_{i,j=1,\dots,n}$ is called a <u>generalized Cartan matrix</u> (GCM) if

(C1) $a_{ii}=2$ for $1\le i\le n$

(C2) $a_{ij}\in\mathbb{Z}_{\le 0}$ for $i\neq j$

(C3) $a_{ij}=0 \Rightarrow a_{ji}=0$

Moreover, it is called a <u>Cartan matrix</u> if

(C4) $A=DB$ for $D$ diagonal, $B$ positive definite

The <u>Dynkin diagram</u> associated to a GCM is a decorated graph with:

(D1) The vertex set is $1, \ldots, n$

(D2) Connect $i \neq j$ with $\max\{|a_{ij}|, |a_{ji}|\}$ lines

(D3) If $i \neq j$ and $|a_{ij}| \geq 2$, add an arrow

$$\underset{i}{\circ} \Leftarrow \underset{j}{\circ}$$

If the Dynkin diagram is a connected, we refer to it as

(C5) $A$ is indecomposable $\square$

<u>Example</u>: (1) Let $(V, \Delta, \Delta^\vee, (\;)^\vee)$ be an abstract root system. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a choice of positive roots. Let $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. Then $A = (a_{ij})_{1 \leq i,j \leq n}$ is a Cartan matrix: (C1)-(C3) are straightforward

to see. To see C(4), pick a $W$-invariant scalar product $( , )$ on $V$. Then one may show that

$$\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \quad \forall \alpha, \beta \in \Delta.$$

Let $B = ((\alpha_i, \alpha_j))_{1 \leq i,j \leq n}$ be the Gram matrix of $( , )$.
and $D = \operatorname{diag}\left(\frac{2}{(\alpha_i,\alpha_i)}\right)_{1 \leq i \leq n}$. Then $A = BD$.

Choosing this, helps us to draw the root system in the Euclidian v.s. $(V, ( , ))$.

Vice versa a Cartan matrix yields a root system!

(2) Let $\mathfrak{g} \supset \mathfrak{h}$ be a s.s. Lie alg. with Cartan.

We saw in 0.10 Thm A, that $(\mathfrak{h}^*_{\mathbb{R}}, \Delta, \Delta^\vee, ( )^\vee)$

yields a root system, and hence a Cartan matrix.

In this case, the scalar product $(\ ,\ )$ can be chosen as the Killing form. $\square$

<u>Exercise</u>: Convince yourself, using the Killing form, that

$$(\alpha^\vee, \beta^\vee) = \sum_{\gamma \in \Delta} \langle \gamma, \alpha^\vee \rangle \langle \gamma, \beta^\vee \rangle$$

Dually,

$$(\alpha, \beta) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma^\vee \rangle \langle \beta, \gamma^\vee \rangle$$

This is an alternative formula to obtain $B$ from $A$. $\square$

Thm A: All indecomposable Dynkin diagrams are of the form

Classical

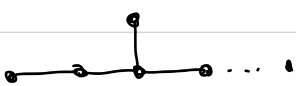
$A_n$ $\quad sl_{n+1}$

$B_n$ $\quad so_{2n+1}$

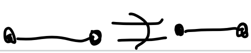
$C_n$ $\quad sp_{2n}$

$D_n$ $\quad so_{2n}$

Exceptional



$E_6, E_7, E_8$



$F_4$

$G_2$

Proof: Omitted

Thm B: The are natural bijection between (isomorphism classes) of

(1) Cartan matrices (2) Dynkin diagrams

(3) (abstract) root systems (4) semisimple Lie algebras.

Proof: Omitted. $\square$

## 0.12. An extended example: $Sp_4$

Consider the symplectic matrix

$$J_4 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

and the associated symplectic Lie algebra

$$\mathfrak{g} = \mathfrak{sp}_4 = \{ X \in \mathfrak{gl}_4 \mid X J_4 + J_4 X^{tr} = 0 \}$$

$$= \left\{ \begin{bmatrix} A & B \\ C & -A^{tr} \end{bmatrix} \;\middle|\; \begin{array}{l} A, B, C \in \mathbb{C}^{2\times 2} \\ B = B^{tr},\ C = C^{tr} \end{array} \right\}$$

which has dimension $4 + 2 \cdot 3 = 10$. We choose the Cartan

$$\mathfrak{h} = \left\{ \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & -x_1 & \\ & & & -x_2 \end{bmatrix} \right\}$$

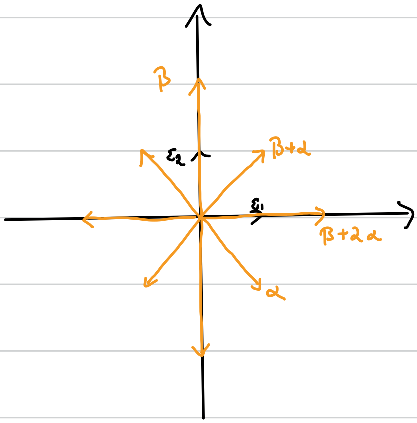
We denote the obvious bases of $\mathfrak{h}$ and $\mathfrak{h}^*$ by $\{\varepsilon_1, \varepsilon_2\}$ and $\{\varepsilon_1^*, \varepsilon_2^*\}$.

We then obtain the roots and coroots:

| $\gamma$ | $\mathfrak{g}_\gamma$ | $H_\gamma$ | $\gamma^\vee$ |
|---|---|---|---|
| $\alpha = \varepsilon_1^* - \varepsilon_2^*$ | $\left[\begin{array}{cc\|cc} & a & & \\ & & & \\ \hline & & & \\ & & -a & \end{array}\right]$ | $\left[\begin{array}{cc\|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right]$ | $\alpha^\vee = \varepsilon_1 - \varepsilon_2$ |
| $\beta = 2\varepsilon_2^*$ | $\left[\begin{array}{cc\|cc} & & & \\ & & & a \\ \hline & & & \\ & & & \end{array}\right]$ | $\left[\begin{array}{cc\|cc} & & & \\ & 1 & & \\ \hline & & & \\ & & & -1 \end{array}\right]$ | $\beta^\vee = \varepsilon_2$ |
| $\alpha+\beta = \varepsilon_1^* + \varepsilon_2^*$ | $\left[\begin{array}{cc\|cc} & & & a \\ & & a & \\ \hline & & & \\ & & & \end{array}\right]$ | $\left[\begin{array}{cc\|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right]$ | $(\alpha+\beta)^\vee = \varepsilon_1 + \varepsilon_2$ |
| $2\alpha+\beta = 2\varepsilon_1^*$ | $\left[\begin{array}{cc\|cc} & & a & \\ & & & \\ \hline & & & \\ & & & \end{array}\right]$ | $\left[\begin{array}{cc\|cc} 1 & & & \\ & & & \\ \hline & & -1 & \\ & & & \end{array}\right]$ | $(2\alpha+\beta)^\vee = \varepsilon_1$ |

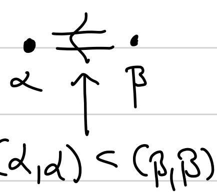
and $\mathfrak{g}_{-\gamma} = (\mathfrak{g}_\gamma)^{tr}$.

The root system has the form (visualised as Euclidian space):




and $\Pi = \{\alpha, \beta\}$ yields a choice of simple roots. The associated Cartan matrix and Dynkin diagram are:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$




$$A = DB \quad \text{for} \quad D = \begin{pmatrix} 1/2 & \\ & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$$

The $\alpha$-string through $\beta$ is of the form

$$\beta,\ \beta+\alpha,\ \underset{-\langle \beta, \alpha^\vee \rangle}{\underset{\|}{\beta+2}}\alpha$$

Correspondingly, we obtain an action of

$$\mathfrak{s}_\alpha = \left\langle e_\alpha = \left[\begin{array}{cc|cc} & 1 & & \\ & & & \\ \hline & & & \\ & & -1 & \end{array}\right],\ h_\alpha = \left[\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & \end{array}\right],\ f_\alpha = e_\alpha^{tr} \right\rangle_{\mathbb{C}}$$

$$= \left\{ \begin{bmatrix} A & \\ & -A^{tr} \end{bmatrix} \;\middle|\; A \in \mathfrak{sl}_2 \right\} \cong \mathfrak{sl}_2$$

on the space

$$V_{\alpha,\beta} = \mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta+\alpha} \oplus \mathfrak{g}_{\beta+2\alpha} = \left\{ \begin{bmatrix} & B \\ & \end{bmatrix} \;\middle|\; B = B^{tr} \right\}$$

The action is given by:

$$\left[\begin{bmatrix} A & \\ & -A^{tr} \end{bmatrix}, \begin{bmatrix} & B \\ & \end{bmatrix}\right] = \begin{bmatrix} & AB+BA^{tr} \\ & \end{bmatrix}.$$

This is exactly the $sl_2$-rep. $S^2(V) = L(-\langle\beta,\alpha^\vee\rangle) = L(2)$.

Similarly. $\langle\alpha+\beta, \alpha^\vee\rangle = 0$ corresponds to the fact that $[s_\alpha, s_{\alpha+\beta}] = 0$

Exercise: Do $so_4$ in the same detail.

Lecture 6

# 2 Kac-Moody Lie Algebras

(C1)-(C3) GCM $\longleftrightarrow$ Kac-Moody algebra

$\cup$

$(C4)_{sym}$ $A = \underset{\text{diag.}}{D}\;\underset{\text{symmetric}}{B}$ $\longleftrightarrow$ symmetrizable —"—

$\cup$

$(C4)_{aff}$ $A = \underset{\text{diag}}{D}\;\underset{\substack{\text{pos. semidefinite} \\ \text{corank} \leq 1}}{B}$ $\longleftrightarrow$ affine —"—

$\cup$

(C4) Cartan matrix $\longleftrightarrow$ semisimple Lie algebra

## 1.0. Some experiments

We will experiment and make sense of

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \rightsquigarrow \hat{\mathfrak{sl}}_2$$

Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials.

We obtain the <u>loop algebra</u>

$$\mathcal{L}sl_2 = sl_2(\mathbb{C}[t, t^{-1}]) = \mathbb{C}[t, t^{-1}] \otimes sl_2$$

where for $x, y \in sl_2$, we have

$$[t^m \otimes x, t^n \otimes y] = t^{m+n} [x, y].$$

In particular $sl_2 = 1 \otimes sl_2 \subset \mathcal{L}sl_2$. If we take

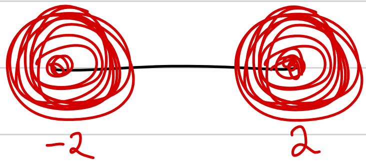
$h_0 = \langle h = 1 \otimes h \rangle_{\mathbb{C}}$ as "Cartan subalgebra", we obtain

a root space decomposition

$$\mathcal{L}sl_2 = \underset{-2}{\mathcal{L} \otimes f} \oplus \underset{0}{h} \oplus \underset{2}{\mathcal{L} \otimes e}$$

and for each $n \in \mathbb{Z}$ an $sl_2$-triple $t^n e, h, t^{-n} f$.

In our "root system", 2, $-2$ appear with multiplicity $\infty$.




To distinguish the many copies of $sl_2$, need to make $\mathfrak{h}_0$ bigger. However, $\mathfrak{h}_0 \subset \mathcal{L}sl_2$ is already "maximal".

<u>Idea</u>: Make $\mathcal{L}sl_2$ bigger!

Define affine $sl_2$ as

$$\tilde{\mathcal{L}}sl_2 = \mathcal{L}sl_2 \oplus \overset{\text{"central"}}{\overset{\downarrow}{\mathbb{C}c}} \oplus \overset{\text{"derivation"}}{\overset{\downarrow}{\mathbb{C}d}}$$

where the Lie bracket is defined by

$$[t^m \otimes x, t^n \otimes y] = t^{m+n}[x,y] + m\,\delta_{m+n,0}\,\overset{\substack{\text{trace form}\\ \downarrow}}{(x,y)}\,c$$

$$[c, \hat{\mathcal{L}}\mathfrak{sl}_2] = 0$$

$$[d, t^n \otimes X] = n\,t^n \otimes X$$

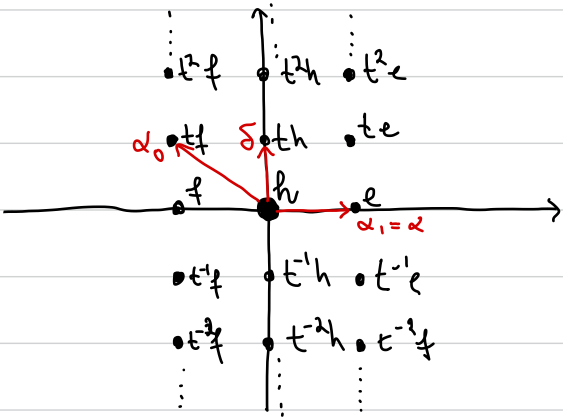
Then, we obtain the bigger Cartan:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d = \langle \alpha_0^\vee = \alpha, c, d \rangle_{\mathbb{C}}$$

We write

$$\mathfrak{h}^* = \mathfrak{h}_0^* \oplus (\mathbb{C}c)^* \oplus (\mathbb{C}d)^* = \langle \alpha = \alpha_1, c^*, \delta = d^* \rangle_{\mathbb{C}}$$

We now obtain the root space decomposition




So we have roots of the form

$$\Delta = \{\pm\alpha + n\delta \mid n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z}\setminus\{0\}\}.$$

We can also construct new coroots. For example, let $\alpha_0 = \delta - \alpha$. Then we get an $\mathfrak{sl}_2$-triple

$$e_0 = t \otimes f \,, \quad f_0 = t^{-1} \otimes e, \quad \alpha_0^\vee = h_0 = [e_0, f_0] = -h + c.$$

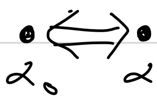
So, we have:

$\alpha_0 = \alpha - \delta$, $e_0 = t \otimes f$, $f_0 = t^{-1} \otimes e$, $\alpha_0^\vee = h_0 = -1 \otimes h + c$

$\alpha_1 = \alpha$, $e_1 = 1 \otimes e$, $f = 1 \otimes f$, $\alpha_1^\vee = h_1 = 1 \otimes h$

and obtain the generalized Cartan matrix

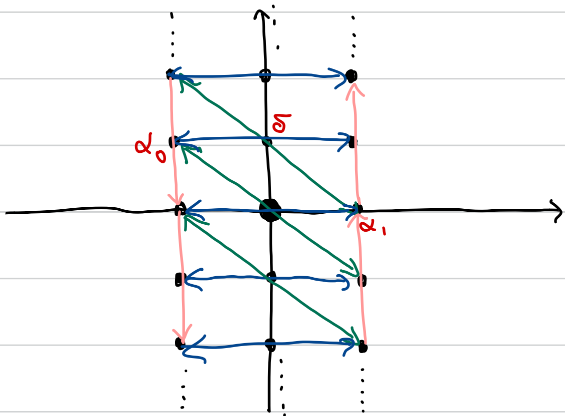
$$\left(\langle \alpha_j, \alpha_i^\vee \rangle\right)_{i=0,1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$




In fact, $e_0, e_1, f_1, f_2$ and $h$ generate $\hat{\mathcal{L}} sl_2$ and one can deduce from the Cartan matrix a system of generators!

We can also study the Weyl group

$$W = \langle s_0, s_1 \rangle$$




Note that $s_1 s_0 = \tau$ acts via the shear:

$$\alpha_1 \mapsto \alpha_1 + 2\delta$$
$$\delta \mapsto \delta$$

So that $W$ is the infinite group

$$W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$$

$$= \langle s_1, \tau \mid s_1^2 = 1,\ s_1 \tau s_1 = \tau^{-1} \rangle = \underset{\langle s_1 \rangle}{\underset{\parallel}{S_2}} \ltimes \underset{\langle \tau \rangle}{\underset{\parallel}{\mathbb{Z}}}$$

lets note some differences to the semisimple case

- The pairing $\langle\, ,\, \rangle$ restricted to $\mathbb{Z}\Delta \times \mathbb{Z}\Delta^\vee$ is degenerate

- There is no scalar product $(\, ,\, )$ on $\mathbb{R}\Delta$, s.t.

$$\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \Rightarrow A \neq DB \text{ for } B \text{ pos. def.}$$

- There a roots, called <u>imaginary roots</u>,

$$\Delta^{im} = \{ n\delta \mid n \in \mathbb{Z} \setminus \{0\} \}$$

which are not <u>real</u>, that is, conjugate to simple roots

$$\Delta^{re} = W\Pi \neq \Delta = \Delta^{re} \uplus \Delta^{im}$$

<u>Exercise:</u> Experiment!

## 1.1. Realizations

Our next goal is to construct $\mathfrak{g}$ from a GCM $A$.

We start with constructing $\mathfrak{h} \subset \mathfrak{g}$.

Def A: A <u>realization</u> of a matrix $A \in \mathbb{C}^{n \times n}$ of rank $\ell$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where

- $\mathfrak{h}$ is a complex vector space,
- $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}^*$

and the following conditions are fulfilled

(R1) $\Pi$ and $\Pi^\vee$ are linearly independent

(R2) $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ji}$

(R3) $n - \ell = \dim \mathfrak{h} - n$ $\square$

**Prop:** There is a natural bijection

$$\{\text{complex square matrices}\}/_{\substack{\text{permutation}\\ \text{of index set}}} \Leftrightarrow \{\text{realization}\}/_{\text{iso}} .$$

**Proof:** "$\Rightarrow$" Let $A \in \mathbb{C}^{n\times n}$ of rank $l$.

We can reorder the indices, such that

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

and $A_1 \in \mathbb{C}^{l \times n}$ has full rank. Now, let

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix},$$

$h = \mathbb{C}^{2n-l}$, $\alpha_1, \dots, \alpha_n$ the first $n$ coordinate vectors in $h$

and $\alpha_1^\vee, \dots, \alpha_n^\vee$ the first $n$ rows of $C$.

"$\Leftarrow$" Given a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$, complete

$\Pi = \{\alpha_1, \dots, \alpha_n\}$ to a basis $\{\alpha_1, \dots, \alpha_{2n-\ell}\}$ of $\mathfrak{h}^*$

Then, we obtain

$$(\langle \alpha_j, \alpha_i^\vee \rangle) = \begin{array}{c|cc|c} & n & n-\ell & \\ \ell & A_1 & B & \ell \\ n-\ell & A_2 & D & n-\ell \end{array}$$

By adding linear combinations of $\{\alpha_1, \dots, \alpha_n\}$ to

$\{\alpha_{n+1}, \dots, \alpha_{2n-\ell}\}$, we can assume $B = 0$.

Since $D$ is invertible, we can replace

$\{\alpha_{n+1}, \dots, \alpha_{2n-\ell}\}$ by a linear combination, so

that $D = I$. Hence, the realisation is isomorphic to the one constructed in "$\Rightarrow$" $\square$

<u>Def B</u>: Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation. Then

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^\vee = \sum_{i=1}^{n} \mathbb{Z}\alpha_i^\vee$$

are called the <u>root</u> and <u>coroot</u> <u>lattice</u>. Moreover, we define

$$Q_+ = \sum \mathbb{Z}_{\geq 0}\alpha_i \subset Q, \quad Q_+^\vee = \sum \mathbb{Z}_{\geq 0}\alpha_i^\vee$$

For $\alpha = \sum k_i \alpha_i$, the height is

$$\mathrm{ht}\,\alpha = \sum k_i$$

and for $\lambda, \mu \in \mathfrak{h}^*$, we write $\lambda \leq \mu$ if

$\mu - \lambda \in Q_+$ $\square$

## 1.2 An auxiliary algebra

Let $A \in \mathbb{C}^{n \times n}$ with realization $(\mathfrak{h}, \Pi, \Pi^\vee)$.

<u>Def</u>: We define $\tilde{\mathfrak{g}}(A)$ as the Lie algebra with generators $e_i, f_i, \mathfrak{h}$ and relations

(R1) $\quad [e_i, f_j] = \delta_{ij} \alpha_i^\vee$

(R2) $\quad [h, h'] = 0 \quad \forall h, h' \in \mathfrak{h}$

(R3) $\quad [h, e_i] = \langle \alpha_i, h \rangle e_i$

(R4) $\quad [h, f_i] = -\langle \alpha_i, h \rangle f_i$.

We denote by $\tilde{n}_+$ (or $\tilde{n}_-$) the subalgebras generated by $e_1, \dots, e_n$ and $f_1, \dots, f_n$ $\square$

Thm: We get

(1) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$

(2) $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are freely generated by $e_1, \dots, e_n$ and $f_1, \dots, f_n$, respectively.

(3) There is a unique involution $\tilde{\omega}$ on $\mathfrak{g}(A)$, such that,

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i, \quad h \mapsto -h$$

(3) There is a root space decomposition

$$\tilde{\mathfrak{g}}(A) = \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \mathfrak{g}_\alpha$$

and $\dim \mathfrak{g}_\alpha < \infty$, $\tilde{\mathfrak{g}}_\alpha \subset \tilde{\mathfrak{n}}_\pm$ for $\alpha \in \pm Q_+ \setminus \{0\}$ $\square$

(4) There is a unique maximal $\mathfrak{r} \subset \mathfrak{g}(A)$, s.t.

$\mathfrak{r}$ is and ideal and $\mathfrak{r} \cap \mathfrak{h} = 0$. It fulfills

$$\mathfrak{r} = (\mathfrak{r} \cap \mathfrak{n}^-) \oplus (\mathfrak{r} \cap \mathfrak{n}^-) \qquad \text{(as ideals)}$$

<u>Proof</u>: <u>Idea</u>: <u>If</u> $\mathfrak{n}^-$ is free, then $\mathcal{U}(\tilde{\mathfrak{n}}_-) = T^\bullet(V)$ for

$V = \bigoplus \mathbb{C} e_i$. Then, we would get a Verma-module-like

representation: $\tilde{M}(\lambda) = \mathcal{U}(\tilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda$.

By the PBW-theorem (0.2 Thm) $\tilde{M}(\lambda) \cong \mathcal{U}(\tilde{\mathfrak{n}}_-) \cong T^\bullet(V)$

as vector spaces. So, we proceed in the opposite direction:

We construct an action of $\tilde{\mathfrak{g}}(A)$ on $T^\bullet(V)$ mimicking

$\tilde{M}(\lambda)$ // Idea.

Let $\lambda \in \mathfrak{h}^*$, $V = \bigoplus \mathbb{C} v_i$. We define an action of $\mathfrak{g}(A)$ on $T(V)$ via:

(a) $f_i(a) = v_i \otimes a$ for $a \in A$

(b) $h(1) = \langle \lambda, h \rangle 1$ and (inductively)

$$h(v_j \otimes a) = -\langle \alpha_j, h \rangle v_j \otimes a + v_j \otimes h(a), \quad a \in T(V)$$

(c) $e_i(1) = 0$ and

$$e_i(v_j \otimes a) = \delta_{ij} \alpha_i^\vee(a) + v_j \otimes e_i(a), \quad a \in T(V)$$

<u>Claim</u>: This defines a representation.

<u>Proof of claim</u>: Need to check relations (R1) – (R4):

(R1) $(e_i f_j - f_i e_i)(a) = e_i(v_j \otimes a) - v_j \otimes e_i(a)$

$$= \delta_{ij}\alpha_i^\vee(a) + v_j \otimes e_i(a) - v_j \otimes e_i(a)$$

$$= \delta_{ij}\alpha_i^\vee(a)$$

(R2) Check that $\mathfrak{h}$ acts diagonally by induction.

(R3) Exercise

(R4) $(hf_j - f_j h)(a) = h(v_j \otimes a) - v_j \otimes h(a)$

$$= -\langle a_j, h\rangle v_j \otimes a + v_j \otimes h(a) - v_j \otimes h(a)$$

$$= -\langle a_j, h\rangle f_j(a)$$

// Claim

(i) Using the relations, it is easy to see that $\mathfrak{g}(A) = \tilde{n}_- + \mathfrak{h} + \tilde{n}_+$.

Now, let $u = n_- + h + n_+ = 0$. Then, by acting on $\mathbb{T}(V)$, we get for all $\lambda \in \mathfrak{h}^*$

$$0 = u(1) = n_-(1) + \langle \lambda, h \rangle = 0 \quad .$$

Since $n_-(1)$ does not depend on $\lambda$, $\langle \lambda, h \rangle = 0$.

The map $\mathfrak{n}^- \to T(V)$, $n \mapsto n(1)$ is a map of Lie algebras $\mathfrak{n}^- \to (T(V), [\,,\,])$ and hence factors as $\mathfrak{n}^- \to \mathcal{U}(\mathfrak{n}^-) \xrightarrow{\varphi} T(V)$. Since $T(V)$ is a free, the inclusion $V \to \mathfrak{n}^-$ yields a map $T(V) \to \mathcal{U}(\mathfrak{n}^-)$ which is inverse to $\varphi$, so $\mathcal{U}(\mathfrak{n}^-) \cong T(V)$. Hence $\mathfrak{n}^- \to T(V)$ is injective and $n_-(1) = 0 \Rightarrow n_- = 0$. Hence (1) follows.

(2) By PBW, $\tilde{n}_-$ is freely generated by $f_1, \dots, f_n$.

(3) Clear, by checking (R1)–(R4).

(4) By (R3) and (R4)

$$\tilde{n}_{\pm} = \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{g}_{\pm\alpha}$$

Moreover $\dim \tilde{g}_{\pm\alpha} \leq n^{|ht\alpha|}$

(5) <u>Claim</u>: Let $V = \bigoplus_{\lambda \in h^*} V_\lambda$ be an $h$-rep. and

$W \subset V$ an $h$-invariant subspace. Then

$$W = \bigoplus_{\lambda \in h^*} W \cap V_\lambda \,.$$

<u>Proof</u>: Write $w \in W$ as $w = \sum_{i=0}^{m} w_i$, where

$\{\lambda_0, \ldots, \lambda_m\} \subset \mathfrak{h}^*$ is a finite set and $w_i \in V_{\lambda_i}$

Choose $h \in \mathfrak{h}$, s.t., $\langle \lambda_i, h \rangle \neq \langle \lambda_j, h \rangle$ for $i \neq j$.

Then we obtain

$$h^j(u) = \sum_{\lambda \in S} \langle \lambda_i, h \rangle^j \, w_i \in W$$

Let $X = (\langle \lambda_i, h \rangle^j)_{i,j=0,\ldots,m}$ Then $X$ is

a Vandermonde matrix and hence invertible, and

$$X^{-1}(h^i(u))_i = (w_i)_i$$

Hence $w_i \in W$ // Claim.

We use this to see that for any ideal $i \subset \mathfrak{g}$

$$i = \bigoplus_{\alpha} \tilde{\mathfrak{g}}_\alpha \cap i$$

Let $r' = \sum_{\substack{i \subset g \\ i \cap h = 0}} i = \sum_{\substack{i \subset g \\ i \cap h = 0 \\ \alpha \neq 0}} \tilde{g}_\alpha \cap i \subset \bigoplus_{\alpha \neq 0} \tilde{g}_\alpha$

So $r' \cap h$. It follows easily that $r = r'$ is the ideal we were looking for. By (1) and the claim, $r = r \cap \tilde{n}_- \oplus r \cap \tilde{n}_+$ as vector spaces. Now $[f_i, r \cap \tilde{n}^+] \subset r \cap (h \oplus \tilde{n}^+) = r \cap \tilde{n}^+$, and hence $[\tilde{g}(A), r \cap \tilde{n}^+] \subset r \cap \tilde{n}^+$. So $r \cap \tilde{n}^\pm \subset \tilde{g}(A)$ are ideals $\square$